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Minimal Sandpiles on Hexagonal Lattice

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We study the minimal recurrent configurations of the Abelian sandpile model on the hexagonal lattice referred to the dynamics of a nonconservative sandpile model. The one-to-one correspondence between these configurations and the set of maximally oriented spanning trees on the triangular sublattice is constructed. We derive the correlation functions in minimal recurrent configurations on a quasi-one-dimensional $2 \times N$ lattice, compare them with correlations for ordinary recurrent configurations, and argue for asymptotic equivalence between them.

KEY WORDS: Self-organized criticality; sandpile models; spanning trees; correlation functions.

1. INTRODUCTION

Sandpile cellular automata were proposed by Bak, Tang and Wiesenfeld (BTW)[1] to explain the ubiquity of $1/f$ noise and fractal structures. Numerous works have shown that sandpiles do indeed drive themselves into a stationary state characterized by self-similarity of spatial fluctuations. However, the original BTW model does hot contain the $1/f$ spectrum for temporal fluctuations of the total mass of the sandpile. Instead, it contains the simple $1/f^2$ power spectrum[2] irrespective of dimension. Later on, several non-conservative sandpile models manifesting the $1/f$ spectrum have been proposed.[3] A non-conservative model which is much closer to the BTW formulation was considered by Bourzutschky and Bennett.[4]

The Bourzutschky and Bennett model (BBM) can be defined on an arbitrary lattice L. Each site $v \in L$ is characterized by an integer z_v which

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is the number of particles or the local height of the sandpile. The particles are added to the system at randomly chosen sites. If z_v exceeds the critical value, it is set to 1 and one particle is transferred to each neighbor of v. The parallel update is assumed. The BBM displays self-organized criticality and $1/f$ behavior. Ali[5] considered this model in one dimension and derived both the properties analytically. Numerical simulations[6] show that the system under the evolution tends to states with low dissipation. As the set of recurrent configurations of BBM coincides with that of the Abelian sandpile model (ASM),[7] the states of low dissipation correspond to recurrent states of ASM having minimal heights. Thus, very likely, the subset of minimal recurrent configurations of the conservative ASM gives a leading contribution to the dynamics of the non-conservative and non-Abelian BBM.

The minimal recurrent configuration of ASM can be naturally defined[8] as a collection of heights $\{z_v\}$, $v \in L$ which is allowed in the critical state but gets forbidden after the change $z_v \to z_v - 1$ for any site of the lattice.

The considerable progress in investigating the steady state of ASM is attained due to the free-fermionic character of the problem pointed by Dhar.[9] In particular, a spanning tree representation was obtained for the set of recurrent configurations[10] which made it possible to get most of spatial characteristics of the critical state.[11-13]

In order to examine the statistical properties of the minimal recurrent configurations (MRC), one needs a similar representation for them. In this paper, we present a construction that gives the one-to-one correspondence between MRC and a subset of spanning trees. To this end, in Section 2 we prove the theorem permitting a different equivalent definition of MRC. In Section 3, we introduce a notion of "oriented spanning trees" and "maximally oriented spanning trees" and find the correspondence between MRC on the hexagonal lattice and maximally oriented spanning trees on the triangular lattice. Section 4 includes a derivation of correlation functions in MRC on a quasi-one-dimensional $2 \times N$ lattice. We compare correlations for MRC and ordinary recurrent configurations and argue for asymptotic equivalence between them.

2. MINIMAL SANDPILES

We consider ASM on the lattice formed by an arbitrary graph G which consists of N vertices and some set of bonds. In this paper, we use the basic definition of a graph, assuming, following Harary,[14] that graphs have no loops and no multiple bonds. A configuration of ASM is a set of integers (heights) z_v associated with each vertex $v \in G$. The perturbation of the states of the model is defined by the following procedure. One selects at random a vertex v and increases its value by 1: $z_v \to z_v + 1$. If z_v exceeds the threshold value z_{vc} (equal to the number $\deg_G(v)$ of bonds connecting v to the other vertices of the graph G), it topples, i.e., $z_v \to z_v - z_{vc}$ and each nearest neighbor of z_v receives 1 particle: $z_{nn} \to z_{nn} + 1$. We introduce an auxiliary vertex ★ connected by one bond with all boundary sites to play the role of a sink of the sandpile.

In a stable configuration $z_v \leqslant z_{vc}$ for all $v \in G$. All stable configurations can be divided into two classes: recurrent configurations that occur an infinite number of times during infinitely long evolution and transient ones having the zero probability of occurrence in the steady state.

Dhar[9] has proposed a constructive definition of the set of recurrent configurations. He showed that these configurations can be characterized by the absence of so-called "forbidden subconfigurations" (FSC) defined as subsets F satisfying

$$z_v \leqslant \deg_F(v) \qquad \text{for all} \quad v \in F$$

To determine if a given configuration contains FSC, Dhar[9] has introduced a recursive procedure called the burning algorithm. This procedure being applied to each stable configuration leads to the one-to-one correspondence between the set of recurrent configurations of the ASM on a graph G and the set of spanning trees on the same graph.[10]

Besides z_v, each site belonging to a recurrent configuration C can be characterized by a supplementary parameter giving a range of possible changes of z_v without converting C into a forbidden configuration. Fix some vertex v of the graph G and consider stable configurations C_m obtained by all possible substitutions $z_v = m (m = 1, ..., M; M = \deg_G(v))$ and leaving all other z_u, $u \in G$, $u \neq v$ of the configuration C unchanged. As in ref. 12, we call the vertex v of the configuration C to be belonging to the class s_k (or being of type k) if the configurations $C_k, C_{k+1}, ..., C_M$ are recurrent while C_{k-1} (for $k > 1$) is forbidden. We say that the height z_v of the site v is minimal, if z_v is equal to the type k of the corresponding vertex v.

Let us introduce the following

Definition. The recurrent configuration C of the ASM on the graph G is called minimal if all its heights z_v are minimal.

It follows from the definition that a minimal configuration C becomes forbidden after decreasing any of the values z_v, $v \in G$.

Let G be a graph and $V(G)$ be the set of its vertices. Suppose that the set $V(G)$ is a disjoint union of two sets $V(G) = V_1(G) \oplus V_2(G)$ such that every bond l of the graph G connects two vertices of different sets (or different colors): $l = \{v_1, v_2\}$, $v_1 \in V_1(G)$, $v_2 \in V_2(G)$. Then, the graph G is called bichromatic.

The minimal configurations of the ASM on a bichromatic graph G have a specific property. Namely, if a recurrent configuration has minimal heights at the vertices of one color, it is necessarily minimal at all other vertices of the graph G. To obtain this result, let us prove the preliminary

Lemma. Let C be a recurrent configuration of the ASM on an arbitrary graph G. Suppose that for some vertex v the height z_v is strictly less than its critical value: $z_v < z_{vc} = \deg_G(v)$. Then, there exists a nearest neighbor u of v such that the configuration C' of heights $z_v(C') = z_v(C) + 1$, $z_u(C') = z_u(C) - 1$, $z_w(C') = z_w(C)$, $w \in G$, $w \neq v$, $w \neq u$, is also a recurrent configuration.

Proof. Let us prove the lemma by contradiction. Let the vertex v have M neighbors $(\deg_G(v) = M)$, $z_v = M - K$, $1 \leqslant K \leqslant M - 1$ and L of them denoted by v_l, $l = 1,..., L$, $1 \leqslant L \leqslant M$ have $z_{v_l} > 1$. Note that $L > K$. Consider the set of stable configurations C_l, $l = 1,..., L$ obtained from the configuration C by transferring one particle to the site v from each vertex v_l, $l = 1,..., L$.

Suppose that every C_l is not recurrent and contains some FSC $F(v_l)$, $l = 1,..., L$. At least $K + 1$ of these FSC's contain the site v. Otherwise, the unification

$$\bigcup_{l=1}^{L} F(v_l) \oplus \bigcup_{m=L+1}^{M} v_m \oplus v$$

of FSC not containing v, neighboring sites v_m, $m = L + 1,..., M$ having $z_{v_m} = 1$, and the vertex v is the FSC in the initial configuration C.

Without loss of generality, denote the neighbors of v corresponding to $K + 1$ FSC containing v by $v_1,..., v_{K+1}$.

Consider the set of FSC $F(v_k)$, $k = 1,..., K + 1$. Since for any configuration $C_k z_v(C_k) = z_v(C) + 1 \geqslant 2$ each of $F(v_k)$ contains at least one of the sites v_i, $i = 1,..., K + 1$, $i \neq k$. Consequently, the set of vertices v_k, $k = 1,..., K + 1$ contains a subset of N sites $2 \leqslant N \leqslant K + 1$ (let these sites be $v_1,..., v_N$) such that every $v_n \in F(v_j)$ for some $j \neq n$, $1 \leqslant n, j \leqslant N$.

Now, consider the set $F = \bigcup_{n=1}^{N} F(v_n)$. For every site v_n, $1 \leqslant n \leqslant N$ $z_{v_n} < \deg_F(v_n)$ because $v_n \in F(v_j)$ for some v_j, $1 \leqslant j \leqslant N$ $j \neq n$. Consequently, the set F is a FSC in the configuration C. We get a contradiction, so the lemma is proved.

Now we can prove the following

Theorem. Let C be a recurrent configuration on a bichromatic graph G with the set of vertices $V(G) = V_1(G) \oplus V_2(G)$. Let all the sites $v \in V_1(G)$ have minimal heights in C. Then, the configuration C is minimal on the whole graph G.

Proof. Suppose that for some recurrent configuration C all the heights of the sites of one color are minimal and for some site $b \in V_2(G)$ of the other color the configuration C' obtained from C by decreasing the height z_b by 1, is recurrent. Then, according to Lemma, there exists a nearest neighbor $a \in V_1(G)$ of the site b such that the configuration $C'': z_a(C'') = z_a(C) - 1, z_b(C'') = z_b(C') + 1 = z_b(C)$ is also recurrent. But this means that the height z_a of the first color is not minimal. This is a contradiction and the theorem is proved.

3. MAXIMALLY ORIENTED TREES

In this section we construct a mapping from the set of minimal sandpile configurations to the set of spanning trees having a special property.

Consider a recurrent sandpile configuration C on a graph G. According to the burning algorithm, we can construct the spanning tree on the graph G corresponding to this configuration. We choose the directedness of the tree edges in such a way that there exists a unique path from each vertex v to the root ★ along the directed edges of the tree. We call the site v_1 the predecessor of v_2 if the path from v_1 to ★ contains the site v_2. The type of any vertex v of the given sandpile configuration, introduced in Section 2, may be calculated by means of its tree representation. Namely, the site v is of type k if the vertex v on the corresponding tree has exactly $k-1$ predecessors among its nearest neighbors.[12] Indeed, consider the site v of the configuration C which belongs to the class s_k, $k \geqslant 1$. The substitution of $z_v = k-1$ converts C into a forbidden configuration C', having the FSC F which contains the site v and exactly $k-1$ nearest neighbors of v. Let us delete the bonds connecting F to the rest of the lattice except one of the bonds connecting v and its nearest neighbor not included in F. Also, we decrease the heights of sites of the lattice having a nearest neighbor in F. The new configuration C'' on the new lattice is burnt by the same burning procedure as the configuration C, which means that C and C'' are represented by the same trees. Moreover, the sites of F burn only after the site v is burnt. As we have chosen the directedness of the tree opposite to the direction of the fire, the vertex v on the spanning tree corresponding to C has exactly $k-1$ predecessors among its nearest neighbors.

If the height z_v at the site v with $\deg_G(v) = m$ is of type k, the configurations $C_k, C_{k+1},..., C_m$ obtained from C by the substitutions $z_v = j$, $j = k,..., m$ are recurrent and correspond to $m - k + 1$ spanning trees, which differ by a position of the directed edge outgoing from v. We can fix some rule for the correspondence of these height configurations and trees and, if necessary, change the order of the global correspondence of sandpile configurations and spanning trees, coming from the burning algorithm, according to this rule. If the graph G is bichromatic, we can define the correspondence in every vertex of one color. It is possible, as for any two vertices of one type the sets of bonds incident to them do not intersect and the rules may be fixed for these sites independently.

Let us now consider the ASM on a hexagonal lattice. This lattice is a bichromatic graph, so we shall consider the two colors of vertices as black and white. We connect all black sites and consider a triangle sublattice formed by these vertices. Let us introduce the orientation on this sublattice assuming that the bonds of each triangle embracing a white site are oriented in the anti-clockwise order.

Consider an arbitrary spanning tree on the oriented triangle lattice with edges directed to the root. For some sites, directions of the tree edges incident to them are opposite to the orientation of the lattice bonds. We try to change each "wrong" tree edge by another one belonging to the same elementary triangle and directed in accordance with the lattice orientation. It can be done each time when the new position of the tree edge does not lead to a closed loop. If for some tree there are no "improvable" edges, we call this tree a maximally oriented spanning tree.

We will show now that the minimal sandpile configurations on the hexagonal lattice can be put into the one-to-one correspondence with maximally oriented spanning trees on the triangle sublattice.

Let us construct the correspondence between the set of recurrent configurations on the hexagonal lattice and the set of spanning trees on the same lattice. Consider the subset of minimal recurrent configurations. According to the theorem, this subset coincides with the set of configurations having minimal heights at the vertices of one color. As we stated above, we can define the correspondence in such a way that the heights in the sites of this color are consistent with the direction of edges of their tree representation. Let us fix the rule of correspondence for these vertices as it is listed in the second column of Fig. 1. The combinations of tree edges for heights 2 and 3 are shown without regard to possible rotations by the angles $2\pi/3$ and $4\pi/3$.

Then, take a spanning tree on the hexagonal lattice corresponding to some minimal recurrent configuration. For each site of the triangular sublattice, let us define the directed edge of a spanning graph according to the

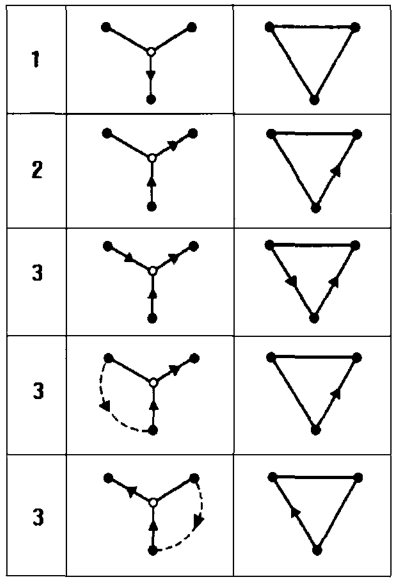


Fig. 1. Correspondence between heights of the minimal sandpile configuration (left column), directed edges of the spanning tree on the hexagonal lattice (central column) and directed edges of the spanning graph on the triangular sublattice (right column). The latter graph is actually a tree (see text).

rules defined in the third column of Fig. 1. We connect by a path of these edges of the elementary triangle each pair of two black vertices connected by a path on the hexagonal lattice inside this triangle. Due to these rules each black site will be supplied by the only arrow on the triangular lattice. A cycle on the triangular sublattice would correspond by construction to a cycle on the hexagonal lattice. Due to the absence of the latter, the spanning graph obtained on the triangular lattice will represent a spanning tree. The direction of all edges of this tree will coincide with the orientation of the sublattice except the only situation shown on the bottom diagrams of Fig. 1. In this case, it is impossible to "improve" the edge of the tree because this will necessarily lead to the appearance of a closed loop. Consequently, the obtained tree on the triangle lattice is a maximally oriented spanning tree. It is clear that different minimal sandpile configurations correspond to different trees; so this mapping is a bijection. Thus, we have proved the following

Theorem. The minimal recurrent configurations on a hexagonal lattice are in the one-to-one correspondence with the maximally oriented spanning trees on the triangle sublattice.

4. CORRELATION FUNCTIONS ON $2 \times N$ LATTICES

Consider the $2 \times N$ hexagonal lattice with periodic boundary conditions in the vertical direction (Fig. 2a). Due to periodicity, it can be viewed as the $2 \times N$ square lattice with closed boundaries (Fig. 2b). The corresponding triangular oriented lattice (Fig. 2c) can be transformed into the simple lattice (Fig. 2d) after identification of opposite vertical bonds in the original hexagonal lattice. Each inclined bond in Fig. 2d corresponds to two bonds directed from a given point to equivalent points in Fig. 2c. Assuming the open boundary to be situated on the right end of the chain, one can construct maximally oriented spanning trees on the resulting directed triangular lattice.

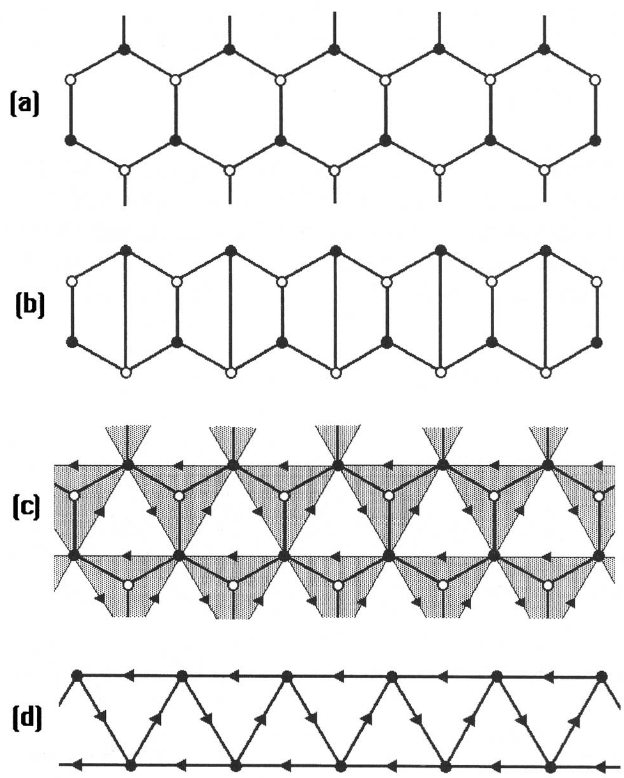
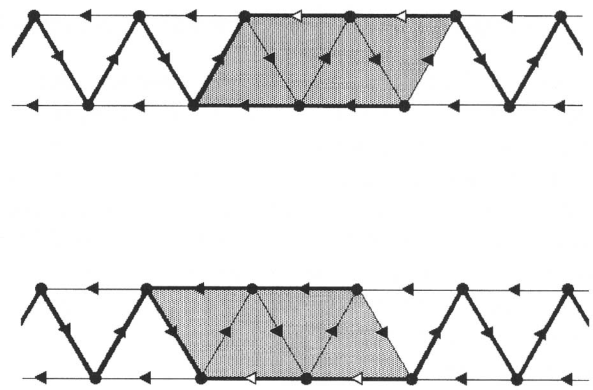


Fig. 2. (a) The $2 \times N$ hexagonal lattice. Two sublattices are marked by black and white circles. (b) The same lattice with periodic boundary conditions in vertical direction. (c) Construction of the correspoding triangular oriented sublattice. (d) The equivalent lattice. Each inclined bond corresponds to two equivalent bonds in Fig. 2(c).

[a]



[b]

Fig. 3. Maximally oriented spanning trees on the equivalent lattice. Direction of all the arrows corresponds to the lattice orientation. The spanning tree edges (bold lines) are directed to the root situated on the right side of the lattice. The white arrows mark edges directed opposite to the lattice orientation. The hatched regions should be contracted to the single inclined bond directed up (a) and down (b).

The obtained trees are zig-zag lines alternating by parallelograms of an arbitrary length (Fig. 3a, b). One of the horizontal sides of such a parallelogram is formed by a branch of a tree having the opposite direction with respect to the lattice orientation. Let z be the statistical weight of a unit of the horizontal length. The generating function of parallelograms is

$$g(z) = 2(z + z^3 + z^5 + \cdots) = \frac{2z}{1 - z^2} \tag{4.1}$$

where the factor 2 is due to duplication of identical bonds and the inclined bonds themselves are considered as parallelograms of the zero width. If one contracts each parallelogram to a single inclined bond, one obtains a new tree having the form of a zig-zag consisting of n inclined bonds. Then, one can write the generation function of maximally oriented spanning trees as

$$G(z) = g(z) + g^2(z) + g^3(z) + \cdots = \frac{g(z)}{1 - g(z)} = \frac{2z}{1 - 2z - z^2} \tag{4.2}$$

The total horizontal length of each tree is N. The number of maximally oriented spanning trees on the $2 \times N$ lattice is given by

$$A_N = \frac{1}{2\pi i} \oint \frac{1}{z^{N+1}} G(z)\, dz \tag{4.3}$$

from which we obtain

$$A_N = \frac{\sqrt{2}}{2}\left[(\sqrt{2}+1)^N + (-1)^N(\sqrt{2}-1)^N\right] \tag{4.4}$$

Now we can calculate the correlation functions in this model. Consider a point i occupied by a single particle ($z_i = 1$). The point i divides the lattice into two parts of the length R and $L(R+L=N)$. The right part must begin with an inclined bond representing a maximally oriented spanning tree half of which is compatible with the condition $z_i = 1$. The left part must end with either a parallelogram or an inclined bond also representing half of proper configurations. The number of left configurations ended with a parallelogram is $A_L - 2A_{L-1}$, the number of left configurations ended with an inclined bond is $2A_{L-1}$. Therefore, the probability $P_i(1)$ that $z_i = 1$ is

$$P(1) = \frac{(A_L - A_{L-1})\, A_R}{2A_N} \tag{4.5}$$

By using Eq. (4.4), for large R, L, N, we get

$$P(1) = \frac{\sqrt{2}-1}{2} \tag{4.6}$$

Analogously, the correlation function $P_{ij}(1, 1)$ of two points i and j with $z_i = 1$ and $z_j = 1$ separated by the distance M can be factorized into three parts corresponding to segments of the lengths R, M, $L(R+M+L=N)$. Using the same arguments as for the derivation of $P(1)$, we can write $P_{ij}(1, 1)$ in the form

$$P_{ij}(1, 1) = \frac{(A_L - A_{L-1})(A_M - A_{M-1})\, A_R}{4A_N} \tag{4.7}$$

which gives, after the substitution of Eq. (4.4), for large M

$$P_{ij}(1, 1) = P(1)\, P(1) + \frac{(-1)^M}{4}(\sqrt{2}-1)^{2M} \tag{4.8}$$

where the interval M between i and j is assumed deeply inside the system.

The height-height correlations for the minimal sandpiles should be compared with those for the ordinary Abelian sandpiles and the directed sandpiles. The recurrent configurations of ASM on the $2 \times N$ hexagonal lattice (Fig. 2a, b) are represented by the spanning trees on this structure

(Fig. 4). The hatched regions correspond to squares having contrawise oriented bonds on the upper and lower sides. As above, we contract the connected hatched squares into a single vertical bond characterized by the generating function:

$$g(z) = 1 + z + z^2 + \cdots = \frac{1}{1-z} \tag{4.9}$$

Then, the generating function of the spanning trees $G(z)$ can be written by means of the auxiliary generating function $S(z)$ of an interval between vertical bonds

$$G(z) = S(z) + S^2(z) + \cdots = \frac{1}{1-S(z)} \tag{4.10}$$

where

$$S(z) = 2g(z)(z + z^2 + \cdots) = \frac{2z}{(1-z)^2} \tag{4.11}$$

The number of spanning trees Λ_N of the horizontal length N is

$$\Lambda_N = \frac{1}{2\pi i} \oint \frac{dz}{z^{N+1}} \frac{2}{1-4z+z^2} \tag{4.12}$$

The denominator in Eq. (4.12) has the roots $\lambda_1 = 2 + \sqrt{3}$ and $\lambda_2 = 2 - \sqrt{3}$. The first of them defines the asymptotics of $\Lambda_N \sim \lambda_1^N$ and their ratio λ_2/λ_1 defines the asymptotics of correlation functions. Therefore, the height-height correlation function of ASM decays asymptotically as

$$P_{ij} \sim (2 - \sqrt{3})^{2M} \tag{4.13}$$

Finally, we consider the directed ASM on the oriented triangular lattice (Fig. 2c).

A spanning tree representing the recurrent configuration for the open boundary situated at the left end of the chain is shown in Fig. 5. The

Fig. 4. A spanning tree on the strip equivalent to the hexagonal lattice $2 \times N$ with periodic boundary conditions shown in Fig. 2(b). All bonds are directed to the root situated at the right side. The hatched region should be contracted to the single vertical bond.

Fig. 5. A. spanning tree on the oriented triangular lattice. Orientation of each bond to the root situated at the left side coincides with the orientation of the lattice.

generating function of oriented spanning trees is quite identical to Eq. (4.2) except the function $g(z)$

$$g(z) = 2(z^3 + z^5 + \cdots) = \frac{2z^3}{1 - z^2} \tag{4.14}$$

which starts with the term $2z^3$ as the minimal distance between kinks is 3.
As a result, one has

$$G(z) = \frac{g(z)}{1 - g(z)} = \frac{2z^3}{1 - z^2 - 2z^3} \tag{4.15}$$

The roots of the denominator in Eq. (4.15) are

$$z_1^{-1} = \lambda_1 = a + b,\ z_2^{-1} = \lambda_2 = \exp(2\pi i/3)\, a + \exp(4\pi i/3)\, b,\ z_3^{-1} = \lambda_2^*,$$

where

$$a = (1 + (\tfrac{26}{27})^{1/2})^{1/3} \qquad \text{and} \qquad b = (1 - (\tfrac{26}{27})^{1/2})^{1/3}$$

Respectively, the number of oriented spanning trees grows as

$$\Lambda_N \sim (a + b)^N \tag{4.16}$$

and correlations deeply inside the directed sandpile decay with the distance M as

$$P_{ij} \sim \left(\frac{|\lambda_2|}{\lambda_1}\right)^M = \left(1 + \frac{a+b}{2}\right)^{-M/2} \tag{4.17}$$

Comparing Eqs. (4.8), (4.13) and (4.17), we see that

$$\left(1 + \frac{a+b}{2}\right)^{-1/2} > (\sqrt{2} - 1)^2 > (2 - \sqrt{3})^2 \tag{4.18}$$

Therefore, the correlation functions of minimal sandpile on the strip are bounded from above and from below by correlations in directed and ordinary sandpiles. One can expect that these inequalities remain valid with broardering of strips. Since both directed and ordinary sandpiles have the same power law asymptotics $1/r^4$ [11] for correlation functions in two dimensions, we can conclude that correlations in two-dimensional minimal sandpiles behave also as $1/r^4$. An answer to the question whether the minimal sandpiles belong to the class of universality of ASM, including exponents of avalanche distributions, seems to be a more difficult problem.

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